§ Recall notations

- $M \simeq \mathbb{Z}^2$, $M^\vee = N$
- Σ : a fan in M
- $T_\Sigma = \langle \Sigma^{[1]} \rangle_{\mathbb{Z}}$

$$\Rightarrow \quad 0 \longrightarrow K_\Sigma \longrightarrow T_\Sigma \xrightarrow{r} M \longrightarrow 0$$

- $R_k = \dfrac{\mathbb{C}[u_1, \cdots, u_k]}{(u_1^2, \cdots, u_k^2)}$

$\mathfrak{m}_R = (u_1, \cdots, u_k)$

§ log derivations

$$\Theta := R_k[T_\Sigma] \otimes_{\mathbb{Z}} N$$

$$\cup$$

$$f \otimes n \sim f \cdot \partial_n$$

Rmk. $n = m^\vee$, then $\partial_n = z^m \dfrac{\partial}{\partial z^m}$

- $f \partial_n (z^m) = f \langle n, r(m) \rangle z^m \quad \forall\, m \in T_\Sigma$
- $\xi \in m_R \cdot \Theta$, define

$$\exp(\xi) \in \mathrm{Aut}(R_k[T_\Sigma])$$

via : $\exp(\xi)(a) = a + \sum_{i=1}^{\infty} \dfrac{\xi^i(a)}{i!}$

this is a finite sum as $m_R^{k+1} = 0$

§ Some special vector fields

- $\forall\, m \in T_\Sigma$ s.t $r(m) \neq 0$,

  choose $n \in r(m)^\perp := \{ n \in N \mid \langle n, r(m) \rangle = 0 \}$

- $I \subset \{1, \cdots, k\}$, $u_I = \prod_{i \in I} u_i$

- $\xi = c \cdot u_I \cdot z^m \partial_n$, $c \in \mathbb{C}$

Check $\xi(z^{m'}) = c\, u_I\, z^m\, \partial_n (z^{m'})$

$$= c\, u_I \langle n, r(m') \rangle\, z^{m'+m}$$

$$\xi^2 (z^{m'}) = 0 \quad \text{as} \quad u_I^2 = 0$$

$$\Rightarrow \exp(\xi)(z^{m'}) = z^{m'} + c\, u_I \langle n, r(m') \rangle\, z^{m'+m}$$

$$= z^{m'} \left(1 + c\, u_I \langle n, r(m') \rangle z^m \right)$$

§ The Lie algebra $a_{\Sigma, k}$.

Lie bracket :

$$[z^m \partial_n , z^{m'} \partial_{n'}] = z^m (\partial_n z^{m'}) \partial_{n'} - z^{m'} (\partial_{n'} z^m) \partial_n$$

$$= z^m \langle n, r(m') \rangle z^{m'} \partial_{n'} - z^{m'} \langle n', r(m) \rangle z^m \partial_n$$

$$= z^{m+m'} \left[\langle n, r(m') \rangle \partial_{n'} - \langle n', r(m) \rangle \partial_n \right]$$

$$= z^{m+m'} \partial_{\langle n, r(m') \rangle n' - \langle n', r(m) \rangle n}$$

Now check :

$$\Big\langle \langle n, r(m') \rangle n' - \langle n', r(m) \rangle n , r(m+m') \Big\rangle$$

$$= \cancel{\langle n, r(m') \rangle \cdot \langle n', r(m) \rangle} + \langle n, r(m') \rangle \langle \underbrace{n', r(m')}_{=0} \rangle$$

$$- \langle n', r(m) \rangle \langle \underbrace{n, r(m)}_{=0} \rangle - \cancel{\langle n', r(m) \rangle \langle n, r(m') \rangle}$$

$$= 0$$

Let $\mathfrak{a}_{\Sigma,k} = \bigoplus_{\substack{m \in T_\Sigma \\ r(m) \neq 0}} m_R \cdot (z^m \otimes r(m)^\perp)$

$\subset \mathbb{H}$

$\Rightarrow$ $\mathfrak{a}_{\Sigma,k}$ is a Lie algebra.

$\Rightarrow$ the corresponding Lie group

$$V_{\Sigma,k} = \{ \exp(\xi) \mid \xi \in \mathfrak{a}_{\Sigma,k} \}$$

$$\subset \mathrm{Aut}(R_k[T_\Sigma])$$

§ $V_{\Sigma,k}$ preserves the symplectic form.

- Choose a basis $M = \langle e_1, e_2 \rangle$

  Set $\Omega = d\log z^{e_1} \wedge d\log z^{e_2}$

$$= \frac{dz^{e_1}}{z^{e_1}} \wedge \frac{dz^{e_2}}{z^{e_2}}$$

- For any $m \in M$, define $X_m \in N$ s.t

$$X_m : M \longrightarrow \wedge M = \mathbb{Z}(e_1 \wedge e_2) \simeq \mathbb{Z}$$
$$m' \longmapsto m \wedge m'$$

What is X_m explicitly?

Suppose $m = a_1 e_1 + a_2 e_2$, then

$$X_m = -a_2 e_1^\vee + a_1 e_2^\vee$$

Check:

$$m \wedge e_1 = a_2(e_2 \wedge e_1) = -a_2(e_1 \wedge e_2)$$
$$m \wedge e_2 = a_1(e_1 \wedge e_2)$$

Thus, Ω defines an isomorphism:

$$M \xrightarrow{\sim} N$$

$$m \longleftrightarrow X_m$$

- Further: $X_m(m) = m \wedge m = 0 \Rightarrow X_m \in r(m)^{\perp}$.

- For $f = z^m$, define

$$X_f = -z^m X_{r(m)} \in \Theta$$

<u>Claim</u> X_f is a Hamitonian vector field

i.e. $\iota(X_f)\Omega = df$.

Indeed, by Cartan's formula, we have

Lie derivative

$$\mathcal{L}_{X_f}\Omega = \underbrace{d(\iota(X_f)\Omega)}_{\substack{= 0 \\ \text{by Claim}}} + \underbrace{\iota(X_f).(d\Omega)}_{\substack{= 0 \\ \Omega \text{ is closed}}} = 0$$

pf of the Claim

Suppose $\gamma(m) = a_1 e_1 + a_2 e_2$

$$\Rightarrow X_m = -a_2 e_1^\vee + a_1 e_2^\vee$$

or $$= -a_2 z^{e_1}\frac{\partial}{\partial z^{e_1}} + a_1 z^{e_2}\frac{\partial}{\partial z^{e_2}}$$

Then :

$$\iota(X_f)\Omega = -z^m \cdot \iota\left(-a_2 z^{e_1}\frac{\partial}{\partial z^{e_1}} + a_1 z^{e_2}\frac{\partial}{\partial z^{e_2}}\right)\frac{dz^{e_1}}{z^{e_1}} \wedge \frac{dz^{e_2}}{z^{e_2}}$$

$$= -z^m\left(-a_2\frac{dz^{e_2}}{z^{e_2}} - a_1\frac{dz^{e_1}}{z^{e_1}}\right)$$

$$= z^m\left(a_2\, d\log z^{e_2} + a_1\, d\log z^{e_2}\right)$$

$$= z^m\, d\log z^{a_1e_1+a_2e_2}$$

$$= z^m\, d\log z^m$$

$$= dz^m \quad \boxtimes$$

Conclusion:

(1) $\mathfrak{a}_{\Sigma,k}$ is generated by $m_R \cdot X_f$.

(2) $V_{\Sigma,k}$ is a group of symplectomorphisms that preseve Ω.

§ The commutator

<u>Lemma</u> If $f \in m_R \cdot R_k[T_\Sigma]$ and $\theta \in V_{\Sigma, k}$, then

$$\theta \circ X_f \circ \theta^{-1} = X_{\theta(f)}$$

<u>Proof</u> Suppose $\theta = \exp(c u_I z^m X_m)$ and $\theta^{-1} = \exp(-c u_I z^m X_m)$.

<u>Recall</u>:

$$\theta(z^{m''}) = z^{m''} \cdot (1 + c u_I \, r(m) \wedge r(m'') \cdot z^m)$$

$$\theta^{-1}(z^{m''}) = z^{m''} \cdot (1 - c u_I \, r(m) \wedge r(m'') z^m)$$

By linearity, may assume $f = z^{m'}$ and check:

$$\Theta \circ X_f \circ \Theta^{-1}(z^{m''})$$

$$= \Theta \circ \left(- z^{m'} X_{r(m')}\right)\left(z^{m''} \cdot \left(1 - c \cdot u_I \cdot r(m) \wedge r(m'') z^{m}\right)\right)$$

$$= -\Theta \circ \left(z^{m'} X_{r(m')}\right)\left(z^{m''} - c u_I r(m) \wedge r(m'') z^{m+m''}\right)$$

$$= -\Theta\Big(r(m') \wedge r(m'') z^{m'+m''}$$

$$- c u_I \left(r(m) \wedge r(m'')\right) \cdot \left(r(m') \wedge r(m+m'')\right) z^{m+m'+m''}\Big)$$

$$(u_I^2 = 0) \quad = -\left(r(m') \wedge r(m'')\right) z^{m'+m''} \cdot \left(1 + c u_I \left(r(m) \wedge r(m'+m'')\right) z^{m}\right)$$

$$+ c u_I \left(r(m) \wedge r(m'')\right) \cdot \left(r(m') \wedge r(m+m'')\right) z^{m+m'+m''}$$

$$= - r(m') \wedge r(m'') z^{m'+m''}$$

$$- c u_I \left(r(m') \wedge r(m'')\right) \cdot \left(r(m) \wedge r(m'+\underline{m''})\right) z^{m+m'+m''}$$

$$+ c u_I \left(r(m) \wedge r(m'')\right) \cdot \left(r(m') \wedge r(m+\underline{m'})\right) z^{m+m'+m''}$$

$$= - r(m') \wedge r(m'') z^{m'+m''}$$

$$- c u_I \left(r(m'+m) \wedge r(m'')\right) \cdot \left(r(m) \wedge r(m')\right) z^{m+m'+m''}$$

On the other hand side

$$X_{\theta(f)}(z^{m''}) = X_{\theta(z^{m'})}(z^{m''})$$

$$= X_{z^{m'}(1+c\cdot u_I\, r(m)\wedge r(m')z^{m})}(z^{m''})$$

$$= \left(-z^{m'}X_{r(m')} - cu_I\, r(m)\wedge r(m')\, z^{m+m'}X_{r(m+m')}\right)(z^{m''})$$

$$= -\, r(m')\wedge r(m'')\, z^{m'+m''}$$

$$-\ cu_I\left(r(m)\wedge r(m')\right)\cdot\left(r(m+m')\wedge r(m'')\right) z^{m+m'+m''}$$

$\blacksquare$

§ A more general setting

- Fix a map $r : P \longrightarrow M$

  P : a toric monoid

  e.g. $P = T_\Sigma \oplus \mathbb{N}^k$
  $\mathbb{C}\langle u_1, \cdots, u_k \rangle$

- log derivations :

  $$\Theta := \Theta(\mathbb{C}[P]) := \mathbb{C}[P] \otimes_{\mathbb{Z}} N$$

  $$\ni$$

  $f \otimes n = f\partial_n$ s.t.

  $$f\partial_n(z^m) = f \langle n, r(m) \rangle z^m$$

- The maximal ideal $\mathfrak{m} \subset \mathbb{C}[P]$ generated by $P \setminus P^*$

  e.g. $P = T_\Sigma \oplus \mathbb{N}^k$, then $\mathfrak{m} = \langle u_1, \cdots, u_k \rangle$

- $I \subset \mathbb{C}[P]$ be a monomial ideal s.t. $\sqrt{I} = \mathfrak{m}$.

  e.g. $I = \langle u_1^2, \cdots, u_k^2 \rangle$

- For any $\xi \in \mathfrak{m} \cdot \Theta(\mathbb{C}[P])$, define $\exp(\xi) \in \mathrm{Aut}(\mathbb{C}[P]/I)$ s.t.

$$\exp(\xi)(a) = a + \sum_{i=1}^{\infty} \frac{\xi^i(a)}{i!}$$

a finite sum mod I.

- The Lie algebra:

$$\mathfrak{a} := \bigoplus_{\substack{m \in \mathfrak{m} \\ r(m) \neq 0}} \mathbb{C} \cdot z^m \otimes r(m)^{\perp} \subseteq \mathfrak{m} \cdot \Theta$$

- The Lie group:

$$V_I := \{ \exp(\xi) \mid \xi \in \mathfrak{a} \}$$

- The completion:

  * $\widehat{\mathbb{C}[P]} := \varprojlim \mathbb{C}[P]/\mathfrak{m}^k$

  * $\widehat{V} := \varprojlim V_{\mathfrak{m}^k}$ a pro-nilpotent subgroup of $\mathrm{Aut}(\widehat{\mathbb{C}[P]})$

§ The scattering diagram.

(1) A ray or a line is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ s.t.

(a) $\mathfrak{d} \subseteq M_{\mathbb{R}}$ is given by $\mathfrak{d} = m_0' - \mathbb{R}_{\geq 0} \cdot m_0$ if $\mathfrak{d}$ is a ray

(b) and $\mathfrak{d} = m_0' - \mathbb{R} \cdot m_0$ if $\mathfrak{d}$ is a line.

where $m_0' \in M_{\mathbb{R}}$, $\underbrace{m_0 \in M \setminus \{0\}}_{\text{integral slope}}$

(c) Let $P_{m_0} := \{ m \in P \mid r(m) \in \mathbb{Q}_{>0} \cdot m_0 \}$

then $f_{\mathfrak{d}} \in \widehat{\mathbb{C}[P]}$ s.t.

- $f_{\mathfrak{d}} = 1 + \sum_{m \in P_{m_0}} c_m z^m$
- $f_{\mathfrak{d}} \equiv 1 \mod \mathfrak{m}$

(2) A <u>scattering diagram</u> $\mathfrak{D}$ over $\mathbb{C}[P]/I$ is a finite collection of lines and rays s.t. $f_{\mathfrak{d}} \in \mathbb{C}[P]$ for each $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}$.

(3) A scattering diagram $\mathfrak{D}$ over $\widehat{\mathbb{C}[P]}$ is a countable collection of lines and rays s.t. only finitely many satisfying

$$f_{\mathfrak{d}} \not\equiv 1 \mod \mathfrak{m}^k$$

for each k.

Notations :

- Support : $\operatorname{Supp} \mathfrak{D} = \bigcup_{\mathfrak{d} \in \mathfrak{D}} \mathfrak{d} \subseteq M_{\mathbb{R}}$
- singularities: $\operatorname{Sing} \mathfrak{D} = \underbrace{\bigcup_{\mathfrak{d} \in \mathfrak{D}} \partial \mathfrak{d}}_{\text{only for rays.}} \cup \bigcup_{\substack{\mathfrak{d}_1, \mathfrak{d}_2 \\ \dim \mathfrak{d}_1 \cap \mathfrak{d}_2 = 0}} \mathfrak{d}_1 \cap \mathfrak{d}_2$

§ γ-ordered product of $\mathfrak{D}$

- A sm immersion $\gamma : [0,1] \longrightarrow M_{\mathbb{R}} \setminus \mathrm{Sing}\,\mathfrak{D}$

  s.t. (1) $\gamma(0), \gamma(1) \notin \mathrm{Supp}\,\mathfrak{D}$

  (2) γ intersects $\mathfrak{D}$ transversally at

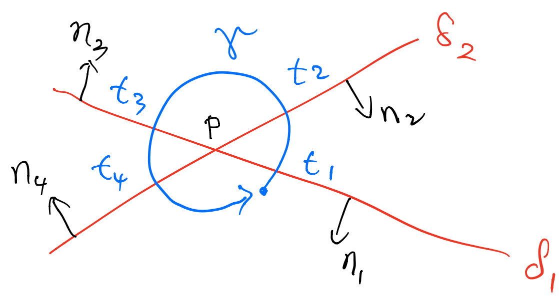
  $$0 < t_1 \leq t_2 \leq \cdots \leq t_s < 1$$

  s.t. (a) $\gamma(t_i) \in \mathfrak{d}_i$

  (b) $\mathfrak{d}_i \neq \mathfrak{d}_j$ if $t_i = t_j$, $i \neq j$

  s is taken as large as possible.

<u>Rmk for (b)</u> There may be several lines or rays with the same support.



- Define :

$$\theta_{\gamma,\delta_i}(z^m) = z^m \cdot f_{\delta_i}^{\langle n_i, r(m)\rangle}$$

for $m \in P$,

and a primitive $n_i \in N$ s.t.

(a) $\langle n_i\ \delta_i \rangle = 0$

(b) $\langle n_i, \gamma'(t_i)\rangle < 0$

Note if $r(m) = 0$ then $\theta_{\gamma,\delta_i}(z^m) = z^m$.

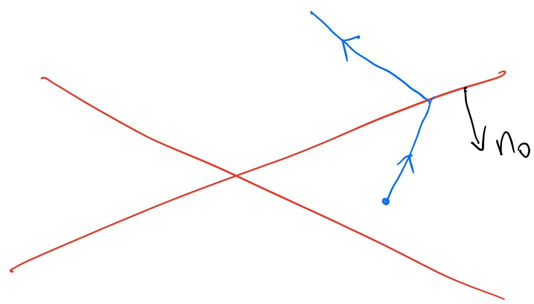
Then define

$$\theta_{\gamma,D} = \theta_{\gamma,\delta_s} \circ \cdots \circ \theta_{\gamma,\delta_1}$$

<u>Some remarks</u> :

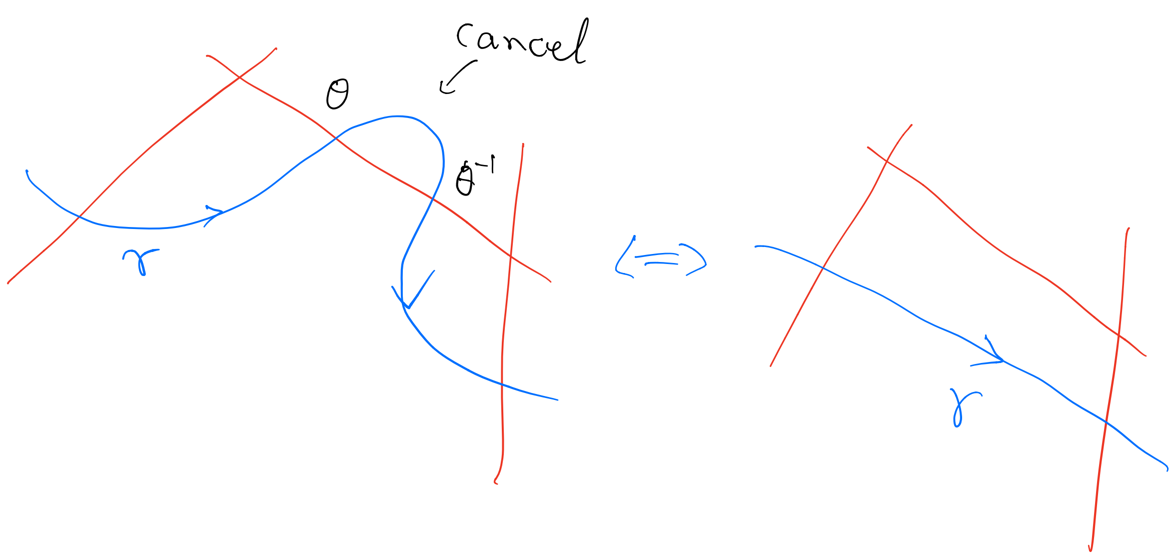
① If γ crosses overlapping lines or rays then the order doesn't matter since they all commute. Note $n_i \in r(m_i)^\perp$

② Allow γ to be piecewise linear at τ_i:



Note that n_0 is primitive.

③ $\Theta_{\gamma, D}$ only depend on the homotopy class of γ:



(4) If D is over $\widehat{\mathbb{C}[P]}$, then

(a) First define

$$\Theta_{\gamma, D} \bmod m^k \in V_{m^k}$$

only cross finite rays and lines

(b) Then define

$$\Theta_{\gamma, D} = \lim \Theta_{\gamma, D} \bmod m^k$$

$$\in \widehat{V}$$

§ Consistency

<u>Thm</u> (1) Suppose $\mathfrak{D}$ is over $\mathbb{C}[P]/I$, Then $\exists$ a scattering diagram $S_I(\mathfrak{D})$ s.t.

(a) $\mathfrak{D} \subset S_I(\mathfrak{D})$

(b) $S_I(\mathfrak{D}) \setminus \mathfrak{D}$ consists of only rays

(c) $\theta_{\gamma, S_I(\mathfrak{D})} = Id \in V_I$

(2) Same is true for $\mathfrak{D}$ defined over $\widehat{\mathbb{C}[P]}$ by taking limit in (1)

<u>Rmk on uniqueness of $S(\mathfrak{D})$</u>

- $S(\mathfrak{D})$ is essentially unique by :

  If $(\mathfrak{d}_1, f_{\mathfrak{d}_1}), \cdots, (\mathfrak{d}_n, f_{\mathfrak{d}_n})$ has the same support, then may replace them by

  $(\mathfrak{d}, \prod_{i=1}^{n} f_{\mathfrak{d}_i})$ where $\mathfrak{d} = \operatorname{Supp} \mathfrak{d}_i$

• By doing this to all overlapping lines and rays, arrive at a new $\mathfrak{D}'$, which is equivalent to $S(\mathfrak{D})$ in the sense:

$$\theta_{\gamma, S(\mathfrak{D})} = \theta_{\gamma, \mathfrak{D}'} \quad \text{in } \hat{V} \ (\text{or } \hat{V}_I).$$

Proof of Thm. Prove by induction:

• Clearly $\theta_{\gamma,\mathfrak{D}} \equiv \mathrm{Id} \mod m^{0+1}$

• Suppose $\theta_{\gamma,\mathfrak{D}} \equiv \mathrm{Id} \mod m^{k+1}$

Set $\mathfrak{D}_n = \mathfrak{D} \mod m^{n+1}$

↳ f_δ may still have higher order terms.

• $\mathfrak{D}'_{k-1} = \{ (\delta, f_\delta) \in \mathfrak{D}_{k-1} \mid f_\delta \not\equiv 1 \mod m^{k+1} \}$

Choose: $p \in \mathrm{Sing}(\mathfrak{D}'_{k-1})$ → this is a finite set.

γ_p: a small loop around p containing no other elements in $\mathrm{Sing}(\mathfrak{D}'_{k-1})$.

- By induction:

$$\theta_{\gamma_p, \mathfrak{D}_{k-1}} \equiv \theta_{\gamma_p, \mathfrak{D}'_{k-1}} \equiv \exp\left(\sum_{i=1}^{s} c_i z^{m_i} \partial_{n_i}\right) \bmod \mathfrak{m}^{k+1}$$

where $m_i \in \mathfrak{m}^k$, $r(m_i) \neq 0$, $n_i \in r(m_i)^{\perp}$ primitive and $c_i \in \mathbb{C}$.

- Let $\mathfrak{D}[p] = \{ (p - \mathbb{R}_{\geq 0} r(m_i), 1 \pm c_i z^{m_i}) \mid i = 1, \cdots, s \}$

where sign is chosen s.t. contribution to $\theta_{\gamma_p, \mathfrak{D}[p]}$ is $\exp(-c_i z^{m_i} \partial_{n_i}) \bmod \mathfrak{m}^{k+1}$

- <u>Key</u>: $\forall\ \xi \in \mathfrak{a} \subset \mathfrak{m} \cdot \Theta$

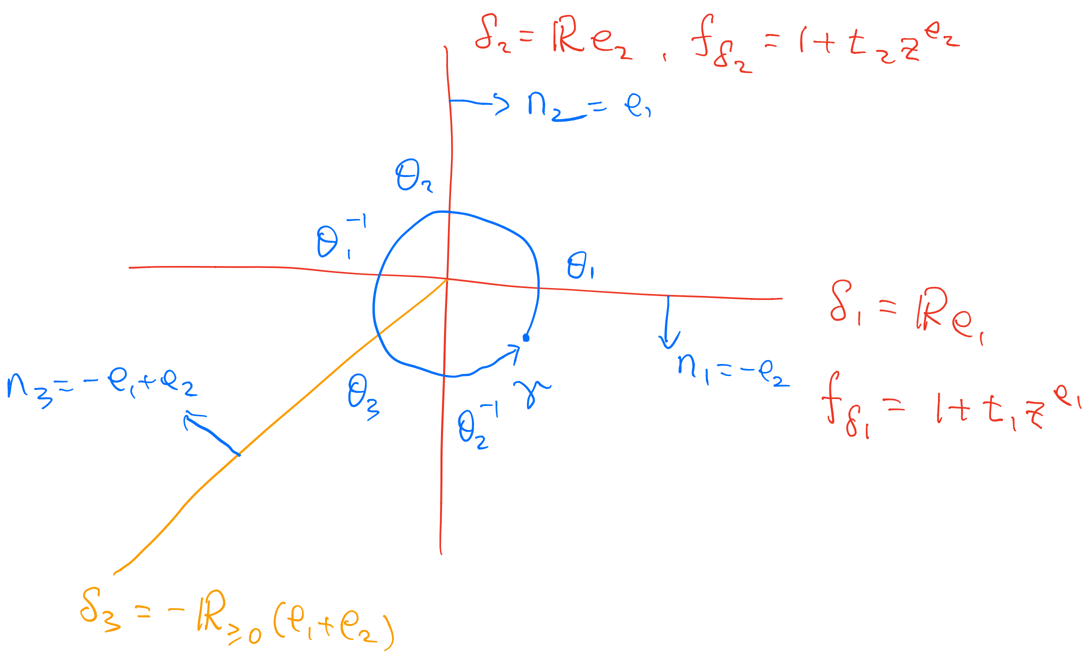
$$[c_i z^{m_i} \partial_{n_i}, \xi] \equiv 0 \bmod \mathfrak{m}^{k+1}$$

$\Rightarrow$ automorphisms induced by rays in $\mathfrak{D}[p]$ (commutes with any automorphisms induced by rays and lines in $\mathfrak{D}_{k-1}$ mod $\mathfrak{m}^{k+1}$)

$$\Rightarrow \quad \Theta_{\gamma_p, \mathfrak{D}_{k+1} \cup \mathfrak{D}[p]} \equiv \mathrm{Id} \mod \mathfrak{m}^{k+1}$$

- Finally take $\mathfrak{D}_k = \mathfrak{D}_{k-1} \cup \bigcup_p \mathfrak{D}[p]$ $\boxtimes$

§ An example



$$P = M \oplus \mathbb{N}^2 = M \oplus \langle t_1, t_2 \rangle$$

$$\mathfrak{m} = \langle t_1, t_2 \rangle$$

$$I = \langle t_1^2, t_2^2 \rangle \subset \mathfrak{m}^2$$

$$\mathfrak{D} = \{(\delta_1, f_{\delta_1}), (\delta_2, f_{\delta_2})\}$$

Goal: $S_I(\mathfrak{D}) = \{(\delta_1, f_{\delta_1}), (\delta_2, f_{\delta_2}), (\delta_3, f_{\delta_3})\}$

$$\Longleftrightarrow \theta_2^{-1} \circ \theta_3 \circ \theta_1^{-1} \circ \theta_2 \circ \theta_1 = Id$$

Note that

$$\theta_i = \exp((\log f_i) \cdot \partial_{n_i})$$

$$(f_i = 1 + g_i) \qquad = \exp(g_i \partial_{n_i})$$

$$\theta_i^{-1} = \exp(-g_i \partial_{n_i})$$

We calculate:

$$z^m \xmapsto{\theta_1} z^m \cdot (1 + t_1 \langle n_1, m \rangle z^{e_1})$$

$$= z^m - t_1 \langle e_2, m \rangle z^{m+e_1}$$

$$\xmapsto{\theta_2} z^m \cdot (1 + t_2 \langle n_2, m \rangle z^{e_2})$$

$$- t_1 \langle e_2, m \rangle z^{m+e_1} \cdot (1 + t_2 \langle n_2, m+e_1 \rangle z^{e_2})$$

$$
\begin{aligned}
&= z^{m} + t_2 \langle e_1, m \rangle z^{m+e_2} - t_1 \langle e_2, m \rangle z^{m+e_1} \\
&\qquad - t_1 t_2 \langle e_2, m \rangle \langle e_1, m+e_1 \rangle z^{m+e_1+e_2}
\end{aligned}
$$

$$
\begin{aligned}
\xmapsto{\theta_1^{-1}} \; & z^{m}\left(1 - t_1 \langle n_1, m \rangle z^{e_1}\right) \\
& + t_2 \langle e_1, m \rangle z^{m+e_2} \cdot \left(1 - t_1 \langle n_1, m+e_2 \rangle z^{e_1}\right) \\
& \left.\begin{aligned} & - t_1 \langle e_2, m \rangle z^{m+e_1} \\ & - t_1 t_2 \langle e_2, m \rangle \langle e_1, m+e_1 \rangle z^{m+e_1+e_2} \end{aligned}\right\} \begin{aligned} &\text{use} \\ &t_1^2 = 0 \end{aligned}
\end{aligned}
$$

$$
\begin{aligned}
&= z^{m} + \cancel{t_1 \langle e_2, m \rangle z^{e_1+m}} + t_2 \langle e_1, m \rangle z^{m+e_2} \\
&\qquad + t_1 t_2 \langle e_1, m \rangle \langle e_2, m+e_2 \rangle z^{m+e_1+e_2} \\
&\qquad \cancel{- t_1 \langle e_2, m \rangle z^{m+e_1}} \\
&\qquad - t_1 t_2 \langle e_2, m \rangle \langle e_1, m+e_1 \rangle z^{m+e_1+e_2}
\end{aligned}
$$

$$
\begin{aligned}
&= z^{m} + t_2 \langle e_1, m \rangle z^{m+e_2} \\
&\qquad + t_1 t_2 \langle e_1 - e_2, m \rangle z^{m+e_1+e_2}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\theta_3}{\longmapsto} z^m\left(1 + t_1 t_2 \langle -e_1+e_2, m\rangle z^{e_1+e_2}\right) \\
&\qquad + t_2 \langle e_1, m\rangle z^{m+e_2} \\
&\qquad + t_1 t_2 \langle e_1 - e_2, m\rangle z^{m+e_1+e_2}
\end{aligned}
\Bigg\}\ \text{use } t_2^2 = 0
$$

$$
= z^m + t_2 \langle e_1, m\rangle z^{m+e_2}
$$

$$
\overset{\theta_2^{-1}}{\longmapsto} z^m \qquad \blacksquare
$$